Three types of outflow dynamics on square and triangular lattices and universal scaling

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In this paper we propose a generalization of the one-dimensional outflow dynamics (KD). The rule is introduced as a simplification of Galam dynamics (GD) proposed in an earlier paper. We simulate three types of outflow dynamics, GD, Stauffer *et al.* dynamics, and KD, both on the square and triangular lattices and show whether the outflow dynamics is sensitive to the lattice structure. Moreover, we took into account several types of initial configuration—random, "stripes," and "circle." We investigate the dependence between the mean relaxation time and the initial density p of up-spins for each type of initial conditions, as well as dependence between the mean relaxation time and the size of the system. As a result, we show differences and similarities between three types of the outflow dynamics.

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I. INTRODUCTION

The outflow dynamics was introduced to describe the opinion change in society. The idea was based on the fundamental social phenomenon called "social validation."

Under the outflow dynamics a system eventually always reaches consensus, like in the famous voter model [1–3]. Several other models describing opinion dynamics were introduced by Deffuant [4], Hegselmann and Krause [5], Krapivsky and Redner [6], and Galam [7].

In this paper, however, we do not focus on social applications of our model (for those interested, reviews can be found in Refs. [8–11]). On the contrary, we investigate here the dynamics from the theoretical point of view.

In this paper we pay particular attention to a generalization of the one-dimensional outflow dynamics to higher dimensions. Several possibilities of such a generalization to the square lattice were proposed by Stauffer *et al.* [12] (see Sec. II) but only some of them were used in the later literature [8–11]. In Ref. [13] we presented comparative studies of the two most interesting generalizations out of all proposed in Ref. [12]. Only slight quantitative differences have been found between these two generalizations.

The outflow dynamics on the triangular lattice were considered only in one paper [14]. In this paper the author studied the generalization of the Sznajd model to the triangular lattice with spreading of mixed opinion and with the pure antiferromagnetic opinion—a pair of two neighboring spins on a triangular lattice influenced its eight neighbors.

Up till now no studies on the influence of the lattice geometry, in the case of regular lattices, for spins endowed with the outflow dynamics were provided. However, the influence of the topology for the relaxation under the outflow dynamics in a case of complex networks has been investigated in Refs. [15–18]. In Ref. [15] the time evolution of the system was studied using different network topologies, starting from different initial opinion densities. A transition from consensus in one opinion to the other was found at the same percentage of initial distribution no matter which type of complex network was used. On the other hand, results presented in Ref. [19] suggest that lattice geometry may influence the network dynamics.

In a broad sense the notion of consensus in a network is a particular case of what can be called coherence or full synchronization between sets of coupled elements, subject to some sort of local dynamics or updating rule [19,20]. In the paper [19] the influence of lattice geometry in network dynamics, using a binary cellular automaton with nearest-neighbor interactions, has been studied. It was shown that geometric structures are more cohesive than others, tending to keep a given initial configuration.

The first general question we pose in this paper is the following: Is the outflow dynamics sensitive to the lattice topology (like in the case of binary cellular automaton on regular networks [19]) or not (as suggested in the case of complex networks [15])? To answer this question we present results for several types of the outflow dynamics on the square and triangular lattices coming from regular studies on the mean relaxation time.

The second question we pose in this paper is connected to the differences between particular forms of the outflow dynamics. As mentioned above, several generalizations [12,14–18] from one to higher dimensions were proposed, but no regular comparative studies were provided. In Ref. [13] we presented comparative studies of two types of the outflow dynamics on the square lattice and we found no qualitative differences. However, we did not check how the results would change with a lattice topology or with the type of initial conditions. To complete this approach we decided here to treat the matter systematically. Moreover, we introduce in this paper one more type of generalization of one-dimensional outflow dynamics into two dimensions, which is a simplification of one of the dynamics studied in Ref. [13]. All three dynamics are simulated both on square and triangular lattices. We start from different initial densities of up-spins in several types of initial conditions. We measure the mean relaxation time as a function of initial densities of up-spins as well as the dependence between the mean relaxation time and the lattice size. We show that in some cases universal scaling laws exist, while in others they do not.

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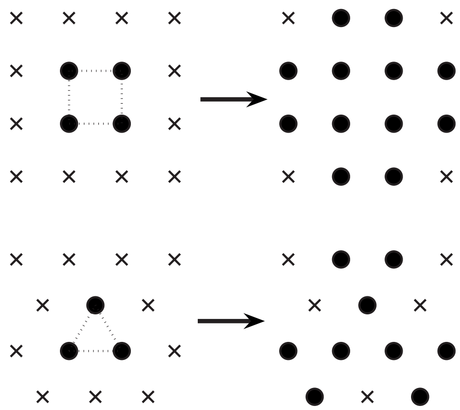

FIG. 1. Stauffer *et al.* dynamical rules of SM on the square (upper panel) and triangular (bottom) lattice. On the square lattice a 2×2 panel of four neighbors (elementary cell) leaves its eight neighbors unchanged, if all four center spins are not parallel. On the triangular lattice a panel of three spins (elementary cell) influences six neighbors along three chains of four spins each, centered about the panel.

II. MODEL

In this paper we consider the generalizations of the one-dimensional outflow dynamics to higher dimensions. Let us begin with recalling the one-dimensional outflow dynamics, described in detail in Ref. [21]. In the original model [22] the pair of neighboring spins S_i and S_{i+1} have been chosen and if $S_iS_{i+1}=1$ the two neighbors of the pair followed its direction, i.e., $S_{i-1}\rightarrow S_i(=S_{i+1})$ and $S_{i+2}\rightarrow S_{i+1}(=S_i)$. Such a rule has been used also in all later papers dealing with the one-dimensional case of the model. However, the case in which $S_iS_{i+1}=-1$ was noted as far less obvious. Several possibilities has been proposed up till now and in general one-dimensional outflow dynamics can be written as [21]:

$$S_i(\tau+1) = \begin{cases} 1 & \text{if } S_{i+1}(\tau)+S_{i+2}(\tau)>0, \\ -S_i(\tau) \text{ with prob } W_0 & \text{if } S_{i+1}(\tau)+S_{i+2}(\tau)=0, \\ -1 & \text{if } S_{i+1}(\tau)+S_{i+2}(\tau)<0. \end{cases} \quad (1)$$

The most known case is for $W_0=0$ and also this case has been generalized into two dimensions. Several possibilities of such a generalization to the square lattice were proposed by Stauffer *et al.* [12]. Six different rules were introduced, but only the following two have been used in later publications: A 2×2 panel of four neighbors leaves its eight neighbors unchanged, if all four center spins are not parallel (see Fig. 1); a neighboring pair persuades its six neighbors to follow the pair orientation if and only if the two pair spins are parallel.

With both these rules complete consensus is always reached as a steady state. Moreover, a phase transition is observed—initial densities below 1/2 of up-spins lead to all spins down and densities above 1/2 to all spins up for large enough systems [12].

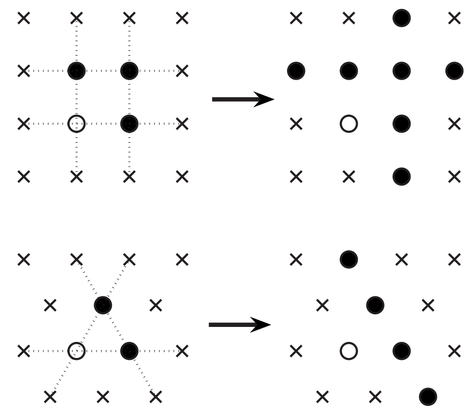

FIG. 2. Galam's dynamical rules of SM on the square (upper panel) and triangular (bottom) lattice. The one-dimensional rule is applied to each of the four (on the square lattice) or three (on the triangular lattice) chains of four spins each, centered about the elementary cell.

Galam (see Stauffer [12]) showed that the updating rule of the one-dimensional SM can be transformed exactly into two dimensions in the following way (see Fig. 2): The one-dimensional rule is applied to each of the four chains of four spins each, centered about two horizontal and two vertical pairs.

In Ref. [13] we compared two rules in which a panel of four spins influenced eight nearest neighbors, i.e., Galam [Galam dynamics (GD)] and the first of Stauffer *et al.* rules [Stauffer *et al.* dynamics (SD)] on the square lattice. This comparison seems to be quite important, since Stauffer *et al.* generalization is more attractive from a social point of view, while the Galam rule is much easier for generalization to other systems (in particular, it was used in the so-called TC model [23,24]). No qualitative difference has been found between these two dynamics.

Here we propose further simplification of the GD—the one-dimensional rule is applied not to each but only one randomly selected chain of four spins (see Fig. 3). The one invented by one of us (G.K.) and introduced here is the dynamics we call KD.

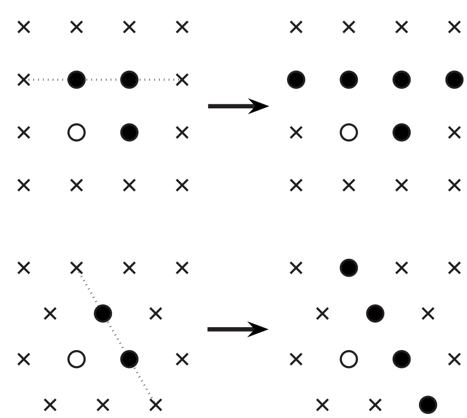

FIG. 3. *K* dynamics of SM on the square and triangular lattice. The one-dimensional rule is applied to only one, randomly selected chain of four spins, centered about elementary cell.

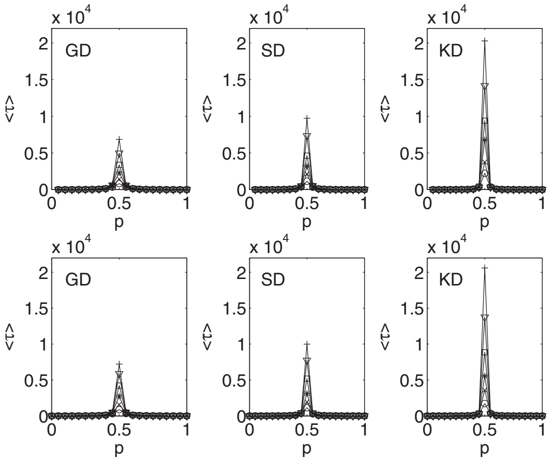

FIG. 4. Comparison of mean relaxation times under three rules (SD, GD, and KD) from a random initial state consisting of p up-spins for two types of two-dimensional lattices—square (upper panel) and triangular (bottom). In each plot results for several lattice sizes, from $L=50$ (◯) to $L=100$ (+), are presented. It is clearly seen that KD for $p=0.5$ is the slowest dynamics on both lattices. Results are averaged over 10^3 samples.

We compare all three dynamics (SD, GD, and KD) on the square and triangular lattices. As in the previous paper [13] we measure the mean relaxation time from an initial state consisting of p up-spins. However, in this paper we consider not only random initial configuration but also two types of ordered initial conditions.

III. RELAXATION TIME FROM RANDOM INITIAL CONDITIONS

We have measured the mean relaxation time from a random initial state consisting of p up-spins for all three types of the outflow dynamics on the square and triangular lattices $L\times L$ using Monte Carlo simulations (we adopted here periodic boundary conditions). We have averaged the relaxation time over 10^3 samples. It should be noticed that in SD (Fig. 1) and GD (Fig. 2) $f_{\max}=8$ spins (on the square lattice) and $f_{\max}=6$ spins (on the triangular lattice) can be changed at maximum in elementary time step, while only two spins can be changed within KD (on both lattices), i.e., $f_{\max}=2$ (see Fig. 3). To compare relaxation times properly we have divided them by $f_{\max}$.

We have found the phase transition for all dynamics—for $p<0.5$ the "all spins up" state is never reached, while for $p>0.5$ this state is obtained with probability 1 (the same result was obtained previously in Ref. [12,13] on the square lattice). Moreover, critical slowing down is observed at $p=0.5$ (see Fig. 4). For $L\to\infty$ we expect the $\delta(0.5)$ function.

It is seen (Fig. 4) that for $p=0.5$, i.e., in the critical point, GD is the fastest dynamics on both lattices, while KD is definitely the slowest one:

$$\tau_{\mathrm{GD}}(0.5) < \tau_{\mathrm{SD}}(0.5) < \tau_{\mathrm{KD}}(0.5). \qquad (2)$$

However from Fig. 4 this is not visible if the relation (4) is valid also outside the critical point. If we look at Fig. 5 we see that for $p<0.5$ the situation is completely reversed and, in general,

$$\tau_{\mathrm{GD}}(p\neq 0.5) > \tau_{\mathrm{SD}}(p\neq 0.5) > \tau_{\mathrm{KD}}(p\neq 0.5). \qquad (3)$$

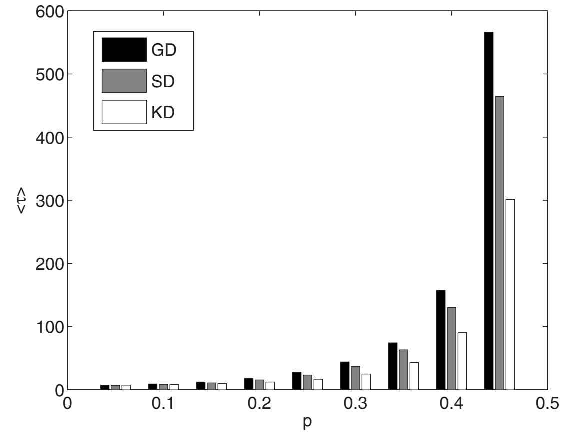

FIG. 5. Comparison of mean relaxation times under three rules (SD, GD, and KD) from a random initial state consisting of $p<0.5$ up-spins for a two-dimensional triangular lattice of 10^4 nodes. It can be seen that KD for $p<0.5$ is the fastest dynamics among all three dynamics. The same result was obtained also for the square lattice. All results are averaged over 10^3 samples.

We should now address a very intriguing question—why is the dynamics which is the slowest in the critical point the fastest outside this point and vice versa? Is it connected somehow to a spatial structure which is created for a different initial concentration p of up-spins? It can be observed that for $p<0.5$ a concentration $c(t)$ of up-spins decreases very fast and after a short time (50–200 MCS) small compact clusters of up-spins are created (Fig. 6). On the contrary, for $p=0.5$ initially concentration of up-spins does not change significantly and only fluctuates around $c(0)=p$ but the system orders and after a short time (50–200 MCS) a large cluster of up-spins is created (Fig. 6)

To check this hypothesis, in the next two sections (i.e., in Secs. IV and V) we investigate the evolution of the system under three outflow dynamics from the following two types of ordered initial conditions. (1) "Stripes:" Initially, the system is divided by the straight border into two horizontal

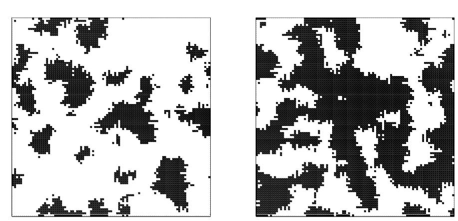

FIG. 6. Configurations of the system under outflow dynamics (type KD) after 200 MCS from a random initial state consisting of $p=0.45$ (left panel, present density of up-spins is 0.2513) and $p=0.5$ (right panel, present density of up-spins is 0.5225) up-spins for a two-dimensional square lattice of 10^4 nodes. The same results are observed for all three dynamics.

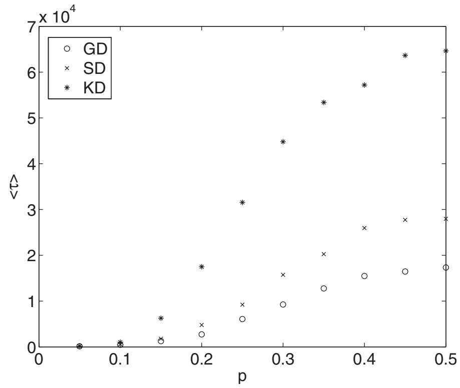

FIG. 7. Mean relaxation times under three rules (SD, GD, and KD) for a two-dimensional $L \times L$ square lattice ($L=100$, i.e., 10^4 nodes). Initially the system is divided by the straight border into two horizontal stripes: pL-width stripe of up-spins and $(1-p)L$-width stripe of down-spins. Results are averaged over 10^3 samples.

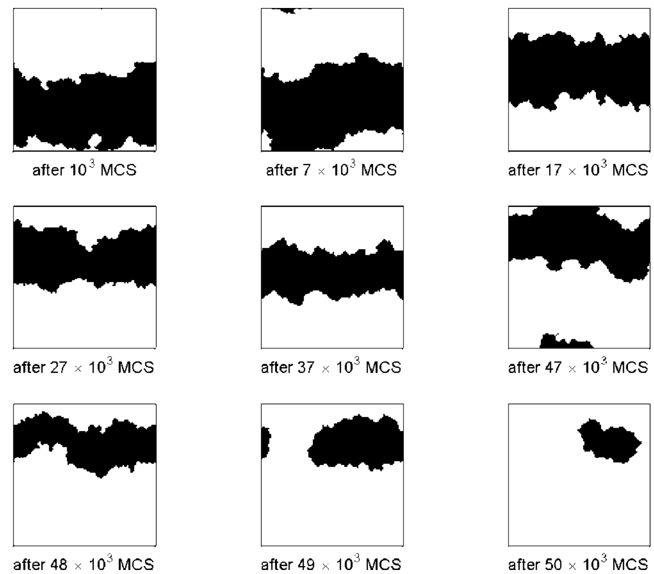

FIG. 8. Snapshots of the sample relaxation under SD on a two-dimensional $L \times L$ square lattice ($L=100$, i.e., 10^4 nodes). Initially the system is divided by the straight border into two equal horizontal stripes. Here $p=0.5$.

stripes—pL-width stripe of up-spins and $(1-p)L$-width stripe of down-spins, i.e., p is again the initial density of up-spins. (2) "Circle:" Initially, a single compact round cluster of up-spins in the middle of the lattice consists of down-spins; p is again the initial density of up-spins.

IV. RELAXATION TIME FROM "STRIPES"

In this section we investigate the relaxation of the system under three types of outflow dynamics from the ordered initial conditions which we call "stripes"—initially the system is divided by the straight border into two horizontal stripes: pL-width stripe of up-spins and $(1-p)L$-width stripe of down-spins, i.e., p is again the initial density of up-spins. For "stripes" no phase transition is observed. Moreover, relaxation under GD is the fastest, while under KD it is the slowest among all three dynamics for all values of initial density of up-spins p (Fig. 7). The same result was obtained for random initial conditions with $p=0.5$, i.e., in the critical point (Fig. 4):

$$\tau_{GD} < \tau_{SD} < \tau_{KD}. \tag{4}$$

As we have seen in the previous section for random initial conditions and $p=0.5$ after a short time a large cluster of up-spins is created (Fig. 6) for all three dynamics. Here we can see that large clusters (stripes) are most unstable under GD and most stable under KD. These results may explain why relaxation from random initial conditions for $p=0.5$ is fastest under GD and slowest under KD.

The second interesting result connected with the relaxation from "stripes" is the lack of the phase transition. However, this could be understood looking at the evolution of the system's configuration. In Fig. 8 snapshots of the sample relaxation under SD on a two-dimensional square lattice is presented. It is seen that relaxation from "stripes" is quasi-one-dimensional in a sense that the structure of the stripes is conserved, although the border between them is no longer straight but rough. Evolution consists of movement of the stripes, roughening the border between them and changing the width of the stripes. Eventually, one of the stripes breaks at one point to form a simply connected cluster and from this moment the evolution leads the system very fast to the final state with all spins in the same state. The same scenario was observed for all three dynamics and for all values of p. It should be mentioned here that "stripes" configuration is the steady state of zero-temperature Glauber dynamics. Several years ago the following question was raised by Spirin *et al.* [28,29]: "*What happens when an Ising ferromagnet, with spins endowed with Glauber dynamics, is suddenly cooled from a high temperature to zero temperature?*" The first expectation was that the system should eventually reach the ground state. However, this is true only for a one-dimensional system. On the square lattice there exist many metastable states that consist of alternating vertical (or horizontal) stripes of widths ≥ 2. These arise because a straight boundary between up and down phases is stable in zero-temperature Glauber dynamics. As we see this is not the case of the outflow dynamics under which the system eventually always reaches the ground state. This result is certainly also a contribution to the discussion about differences between inflow (zero-temperature Glauber) and outflow dynamics (see [21] and references therein).

Thus the lack of the phase transition from "stripes" can probably be explained by the absence of the phase transition in one-dimensional outflow dynamics described by the formula (1) (see also [21]). In Fig. 9 the mean relaxation times from a random initial state consisting of p up-spins for outflow dynamics in one dimension with $W_0=0$ is presented for several lattice sizes. The case of W_0 is consistent with definitions of our two-dimensional dynamics, i.e., under one-dimensional outflow dynamics the pair of neighboring spins S_i and S_{i+1} is chosen and if $S_iS_{i+1}=1$ then the two nearest neighbors of the pair follow its direction. It is seen that no

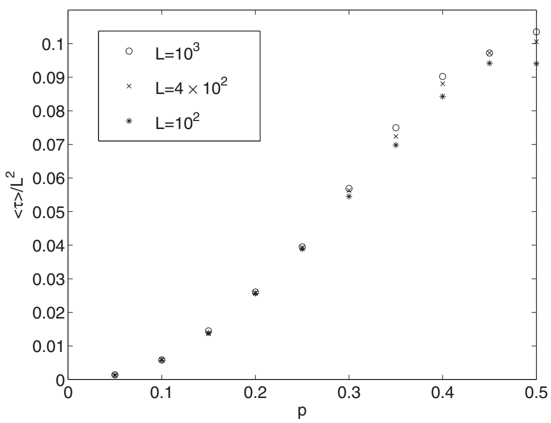

FIG. 9. Mean relaxation times from a random initial state consisting of p up-spins for outflow dynamics in one dimension are presented for several lattice sizes. Under one-dimensional outflow dynamics the pair of neighboring spins S_i and S_{i+1} is chosen and if $S_iS_{i+1}=1$, then the two nearest neighbors of the pair follow its direction. It is clearly visible that in this case the mean relaxation time scales with the lattice size as $\sim L^2$ analogous to the voter model [1–3]. The results presented on the plot are averaged over 10^4 samples.

phase transition is observed. Moreover, the mean relaxation time τ perfectly scales with the size of the system L as $\tau \sim L^2$ for all p. The same scaling law has been obtained already for other one-dimensional consensus dynamics like zero-temperature Glauber dynamics or voter model and can be calculated analytically [1–3].

The similarity between relaxation under one-dimensional dynamics and relaxation under outflow dynamics from "stripes" in two dimensions suggests the existence of a similar scaling law between the mean relaxation time $\langle \tau \rangle$ and the size $N=L\times L$ of the system also in two dimensions. The mean relaxation times from "stripes" consisting of p up-spins for several lattice sizes are presented in Fig. 10. It was obtained that the relaxation time can be scaled with the system's size for all three dynamics with the same scaling exponent $\tau \sim L^a$, $a \approx 3.5$ (see Fig. 10).

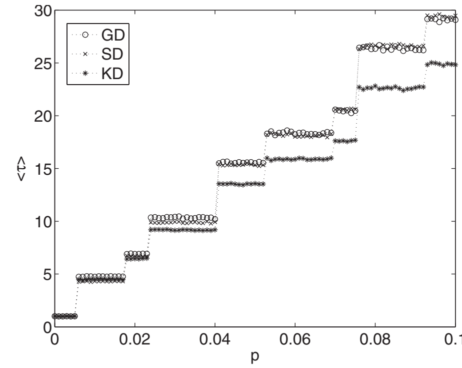

FIG. 11. Mean relaxation times under three rules (SD, GD, and KD) for a small two-dimensional $L\times L$ triangular lattice ($L=25$, i.e., 625 nodes). Initially there is a single compact round cluster of up-spins of radius $R \approx L\sqrt{p/\pi}$ (the value of $p=0.1$ corresponds here to the radius $R=\sqrt{19}\approx 4.36$). The results are averaged over 10^3 samples.

V. RELAXATION TIME FROM "CIRCLE"

In this section we briefly present the results for the relaxation of the system under three types of outflow dynamics from the ordered initial conditions which we call "circle"—initially there is a single compact round cluster of up-spins in the middle of the lattice consisting of down-spins. As we have seen in Fig. 6 starting from random initial conditions the evolution after short times creates small compact isolated clusters. On the other hand, it was observed that for random initial conditions and $p \neq 0.5$ relaxation under KD is fastest, while under GD it is slowest among all three dynamics. Simulations from the "circle" type of initial conditions can help in understanding this relation [see Eq. (3)].

In Fig. 11 we present the mean relaxation times under three rules (SD, GD, and KD) for small two-dimensional $L\times L$ triangular lattice ($L=25$, i.e., 625 nodes) in the case of "circle" initial conditions consisting of pL^2 up-spins (i.e., p is again density of up-spins). It can be seen that in this case we have

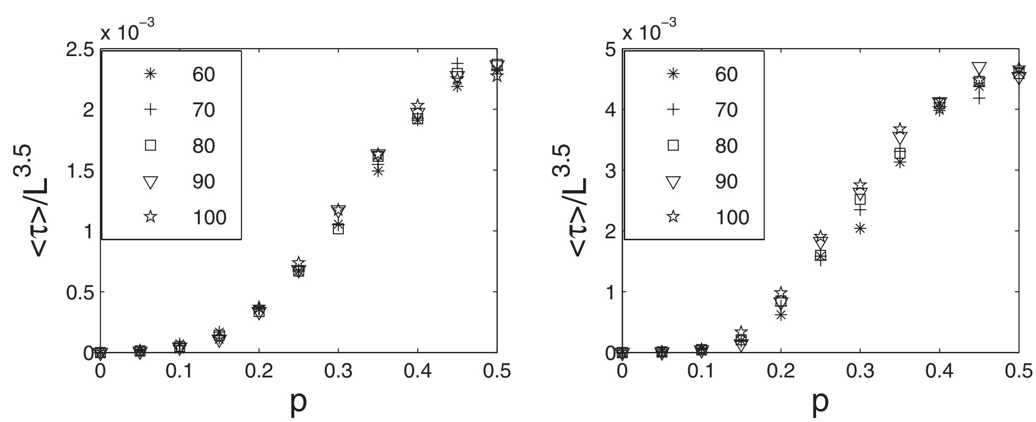

FIG. 10. Mean relaxation times for a two-dimensional $L\times L$ triangular lattice under SD (left panel) and KD (right panel). Initially the system is divided by the straight border into two horizontal stripes: pL-width stripe of up-spins and $(1-p)L$-width stripe of down-spins. In this case the relaxation time scales as $\sim L^{3.5}$ for all three dynamics. The results are averaged over 10^3 samples.

$$\tau_{KD} < \tau_{SD} \approx \tau_{GD}. \quad (5)$$

This behavior is similar to the case of random initial conditions with $p \neq 0.5$, where KD was also the fastest one.

This result shows that small round clusters are more stable under GD, contrary to infinite clusters (like stripes) which are most stable for KD. Summarizing results for random, "stripes," and "circle" initial conditions, we obtain the following:

$$\tau_{KD} > \tau_{SD} > \tau_{GD}$$

"stripes" for all p and random initial conditions for $p=0.5$,

$$\tau_{KD} < \tau_{SD} < \tau_{GD}$$

random initial conditions for $p=0.5$, and

$$\tau_{KD} < \tau_{SD} \approx \tau_{GD}$$

"circle" for p investigated.

VI. IS THE SCALING UNIVERSAL?

It has been found both analytically and numerically that dependence between the mean relaxation time τ and the size of the system L can be expressed by a simple scaling law $\tau \sim L^2$ in the case of a one-dimensional voter model [1–3]. The same scaling is valid also for relaxation in one dimension under zero-temperature Glauber (inflow) dynamics as well as outflow dynamics (see Fig. 9).

In two dimensions a situation is much more complicated. It was found that for a two-dimensional voter model from random initial conditions and $p=0.5$ the following scaling law is valid: $\tau \sim N \log N$ [1–3]. However, this scaling law is valid neither for two-dimensional inflow nor outflow dynamics. It was observed [27–31] that for the Ising ferromagnet with spins endowed with zero-temperature Glauber dynamics there exist many metastable states that consist of alternating vertical (or horizontal) stripes of widths ≥ 2. If we start from random initial conditions and let the system evolve under inflow dynamics, we eventually reach the final "stripes" configuration in 1/3 of the simulations [28]. Because a straight boundary between up and down phases is stable in zero-temperature Glauber dynamics we will never leave such a "stripe" state—for this reason the mean relaxation time is infinite. As we have seen in previous sections (Secs. III and IV), this is not the case for outflow dynamics under which the system eventually always reaches the ground state.

The question is whether the scaling law obtained for the two-dimensional voter model is valid in the case of outflow dynamics. Up till now we have found the scaling law for systems endowed with outflow dynamics initially ordered in "stripes" configuration (see Fig. 7). In this case the mean relaxation time τ scales with the system size $N=L\times L$ as $\tau \sim Ł^{3.5}$ for all three outflow dynamics both on the square and triangular lattice. However, this scaling is not valid in a case of random initial conditions. It occurs that for random initial conditions with the density p of up-spins we can find p-dependent scaling laws: $\tau \sim L^{a(p)}$ (see Fig. 12).

As we see for consensus dynamics with binary variables, scaling laws are universal in one dimension. It should be mentioned here that all these results have been obtained in the case of random sequential updating. It would be interesting for future work to check whether the same scaling is obtained for other types of updating such as, e.g., synchronous or c-synchronous updating [21].

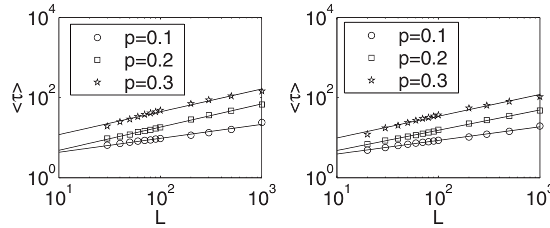

FIG. 12. Scaling of the mean relaxation time with the system size for the Stauffer *et al.* rule. Initial state consists of p randomly distributed up-spins for two types of two-dimensional $L\times L$ lattices—square (left panel) and triangular (right panel). Similar scaling is observed also for GD and KD. It is visible that the scaling exponent for random initial conditions is p-dependent. Results are averaged over 10^3 samples and the largest simulated lattice consists of $N=10^3\times 10^3=10^6$ nodes.

Contrary to one dimension, even within the outflow dynamics no single scaling law can be found in two dimensions—it depends strongly on the initial configuration of the system. However, a very intriguing result connected with scaling can be obtained if we look at the distribution of relaxation times instead of mean relaxation time alone.

VII. DISTRIBUTION OF RELAXATION TIMES

In the mean field approach [25] and in a one-dimensional system it has been found that the distribution of waiting times has an exponential tail with a p-independent exponent. Results for the square lattice for SD and GD were presented in Ref. [13]. Under both dynamics the distribution of relaxation times has an exponential tail, but the exponent is p-dependent. Interestingly, the dependence between the exponent and the initial number of up-spins is identical for both dynamics. It should be mentioned here that in Ref. [12] it was shown that for $p=0.5$ the distribution of relaxation times deviates from the log-normal distribution for SD. However, they plotted a histogram (i.e., an estimate of the probability distribution function) instead of the cumulative distribution function (CDF) and presented it in the log-log scale. In Ref. [13] to compare our results with the results obtained in Ref. [12] we calculated both the cumulative distribution function (in fact, the tail $1-\text{CDF}$) and the histogram of relaxation times. It occurs that our results agree with those presented in Ref. [12]. Already in Ref. [13] the deviation from single exponential decay has been visible. However, for large relaxation times exponential decay for both the histogram and the cumulative distribution function tail was observed in agreement with the results obtained by Slanina and Lavicka for the complete graph [25] and with Schulze [26] who got an exponential decay on the square lattice by introducing both local and global interactions.

In this paper we will not present the histogram of the relaxation times. Instead we focus only on the tail of the

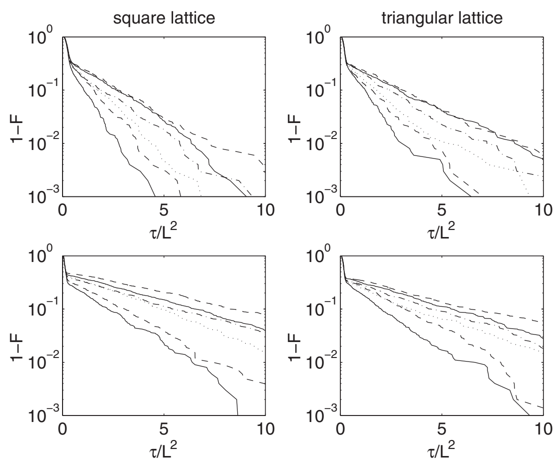

FIG. 13. Tail of the cumulative distribution function of the relaxation time $1-F$ (where F denotes *CDF*) vs τ/L^2 from a random initial state consisting of $p=0.5$ up-spins for SD (top panels) and KD (bottom panels) in a semilog scale are presented. The lattice size runs from $L=50$ (lowest curve) to $L=100$ (uppermost curve). Two regimes—short and long time—are visible for both dynamics in the case of square and triangular lattices. For the short time regime all curves collapse to a single line if we divide the relaxation time τ by the lattice size $N=L^2$. Analogous results are obtained for GD.

cumulative distribution function $1-\text{CDF}$. We would like to explain here our choice and persuade that such a choice gives much more reliable results in estimating distributions. Usually, a histogram is used, because such a representation is much more intuitive. However, such a representation, contrary to CDF, is not one-valued because we are free to choose the number of intervals to which we divide all results. It can be seen very often that the same results look different just because of this not one-valued choice. Moreover, a statistics (i.e., number of results that are represented by one point) in a case of histogram is worse than in a case of CDF, which is clearly visible on plots—in a case of CDF the plot is much smoother. The last reason for which we choose CDF is the following: The histogram is only an estimation of the probability distribution function. For all these reasons we decided to focus on CDF.

In all three dynamics short and long time regimes are observed. These two time regimes are much more visible if we divide relaxation times by the lattice size (see Fig. 13), i.e., we plot the tail the cumulative distribution function of relaxation time $1-\text{CDF}$ versus τ/L^2 instead of τ. Interestingly, results for the short time regime scale with the lattice size with a simple exponent 2. The same exponent is valid for all three dynamics on both square and triangular lattices. It should be mentioned here that for one-dimensional outflow (as well as inflow) dynamics curves for all lattice sizes collapse to a single line if we divide the relaxation time by L^2; this result agrees with the scaling of the mean relaxation time with lattice size $\tau \sim L^2$.

The result obtained from the distribution for the relaxation time is very intriguing and certainly needs deeper investigation which we leave for a future work.

VIII. CONCLUSIONS

In this paper we proposed a generalization of the one-dimensional outflow dynamics (KD). The rule was introduced as a simplification of Galam dynamics (GD) proposed in Ref. [12]. In a previous paper [13] we compared the relaxation from a random initial state consisting of p up-spins under two outflow dynamics on the square lattice [Stauffer *et al.* (SD) [12] and GD]. Here, similar to the previous paper, we have investigated the mean relaxation time from an initial state consisting of p up-spins. However, in this paper we simulated all three types of outflow dynamics, GD, SD, and KD, both on the square and triangular lattices. Moreover, we took into account several types of initial configuration—random, "stripes," and "circle."

Simulation results showed that the relaxations on both lattices (square and triangular) are identical for all three outflow dynamics contrary to results obtained for two-states cellular automaton [19] but in agreement with the results for outflow dynamics on various complex networks [15].

We have found the phase transition for all dynamics—for $p<0.5$ the "all spins up" state is never reached, while for $p>0.5$ this state is obtained with probability 1 (the same result was obtained previously in Refs. [12,13] on the square lattice). Interestingly, in the critical point, GD is the fastest dynamics and KD is definitely the slowest, while outside of the critical point the situation is reversed. We have addressed a very intriguing question—why is the dynamics which is the slowest one in the critical point the fastest one outside this point and vice versa? We connected this behavior with a spatial structure which is created for different initial concentrations p of up-spins—for $p<0.5$ small compact isolated clusters are created, while for $p=0.5$ an infinite cluster is occurring. Starting from two types of ordered states, we have shown that small round clusters are most stable under GD contrary to infinite clusters (like stripes) which are most stable for KD. Summarizing results for random, "stripes," and "circle" initial conditions, we have obtained the following:

$$\tau_{\text{KD}} > \tau_{\text{SD}} > \tau_{\text{GD}}$$

"stripes" for all p and random initial conditions for $p=0.5$,

$$\tau_{\text{KD}} < \tau_{\text{SD}} < \tau_{\text{GD}}$$

random initial conditions for $p=0.5$, and

$$\tau_{\text{KD}} < \tau_{\text{SD}} \approx \tau_{\text{GD}}$$

"circle" for p investigated.

Another interesting result has been obtained while looking at the scaling laws. Both the analytic and numerical approaches in the case of the one-dimensional voter model [1–3] lead to the conclusion that dependence between the mean relaxation time τ and the size of the system Ł can be expressed by a simple scaling law $\tau \sim L^2$. The same scaling is also valid for relaxation in one dimension under zero-temperature Glauber (inflow) dynamics as well as outflow dynamics. On the contrary, for two dimensions even within the outflow dynamics no single scaling law can be found—it depends strongly on the initial configuration of the system.

Probably the most intriguing, yet still preliminary, result presented in this paper is connected with the distribution of relaxation times. For one-dimensional outflow (as well as inflow) dynamics curves for all lattice sizes collapse to a single line if we divide the relaxation time by L^2; this result agrees with the scaling of the mean relaxation time with lattice size $\tau \sim L^2$. In the case of a two-dimensional system in all three dynamics a short and a long time regime in the distribution of relaxation times are observed. These two time regimes are much more visible if we divide relaxation times by the lattice size, i.e., we plot the tail of the cumulative distribution function of relaxation time $1-\mathrm{CDF}$ versus τ/L^2 instead of τ. Interestingly, the results for the short time regime scale with the lattice size with the same simple exponent 2 as obtained for one-dimensional systems. These interesting results certainly require further investigation.

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